

BIJECTIONS FOR ENTRINGER FAMILIES

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ABSTRACT. André proved that the number of down-up permutations on $\{1, 2, \dots, n\}$ is equal to the Euler number E_n . A refinement of André's result was given by Entringer, who proved that counting down-up permutations according to the first element gives rise to Seidel's triangle $(E_{n,k})$ for computing the Euler numbers. In a series of papers, using generating function method and induction, Poupard gave several further combinatorial interpretations for $E_{n,k}$ both in down-up permutations and increasing trees. Kuznetsov, Pak, and Postnikov have given more combinatorial interpretations of $E_{n,k}$ in the model of trees. The aim of this paper is to provide bijections between the different models for $E_{n,k}$ as well as some new interpretations. In particular, we give the first explicit one-to-one correspondence between Entringer's down-up permutation model and Poupard's increasing tree model.

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1. INTRODUCTION

The *Euler numbers* E_n are defined by the generating function

$$\begin{aligned} \sum_{n \geq 0} E_n \frac{x^n}{n!} &= \tan(x) + \sec(x) \\ &= 1 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + 61\frac{x^6}{6!} + 272\frac{x^7}{7!} + 1385\frac{x^8}{8!} + \cdots \end{aligned}$$

Let \mathcal{DU}_n be the set of *down-up permutations* of $[n] := \{1, 2, \dots, n\}$, that is, the permutations $\pi = \pi_1 \pi_2 \dots \pi_n$ on $[n]$ satisfying $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \dots$. For example, the down-up permutations of $[4]$ are:

$$2143, \quad 3241, \quad 3142, \quad 4231, \quad 4132.$$

André [And79] proved that the cardinality of the set \mathcal{DU}_n equals the Euler number E_n . Counting the down-up permutations according to the first term leads to the *Entringer numbers* [Ent66]. More precisely, let $\mathcal{DU}_{n,k}$ be the set of permutations $\pi \in \mathcal{DU}_n$ such that $\pi_1 = k$ and $E_{n,k}$ the cardinality of $\mathcal{DU}_{n,k}$. The first values of $E_{n,k}$ are given in Table 1.

$n \setminus k$	1	2	3	4	5	6	7
1	1						
2	0	1					
3	0	1	1				
4	0	1	2	2			
5	0	2	4	5	5		
6	0	5	10	14	16	16	
7	0	16	32	46	56	61	61

TABLE 1. The first values of Entringer numbers $E_{n,k}$

Theorem 1.1 (Entringer). *The numbers $(E_{n,k})$ ($n \geq k \geq 1$) are defined by*

$$E_{1,1} = 1, \quad E_{n,1} = 0 \ (n \geq 2), \quad E_{n,k} = E_{n,k-1} + E_{n-1,n+1-k}. \quad (1)$$

Iterating the above recurrence, we get $E_{n+1,n+1} = E_{n,n} + E_{n,n-1} + \dots + E_{n,1}$, which is equal to E_n by André's result. Hence the Euler numbers $E_n = E_{n+1,n+1}$ are the diagonal entries in Table 1. As an historical remark, Entringer's recurrence (1) is just a combinatorial interpretation of the Seidel's scheme [Sei77] to compute Euler numbers, i.e.,

$$\begin{array}{cccccccccccccccc} & & & & E_{1,1} & & & & & & & & & & & & 1 \\ & & & & \rightarrow & & & & & & & & & & & & 0 \\ & & & E_{2,1} & & E_{2,2} & & & & & & & & & & & 1 \\ & & & \leftarrow & & \leftarrow & & & & & & & & & & & 0 \\ & & E_{3,3} & & E_{3,2} & & E_{3,1} & & & & & & & & & & 1 \\ & & \leftarrow & & \leftarrow & & \leftarrow & & & & & & & & & & 1 \\ E_{1,1} & \rightarrow & E_{4,2} & \rightarrow & E_{4,3} & \rightarrow & E_{4,4} & & & & & & & & & & 0 \\ & \leftarrow & \leftarrow & & \leftarrow & & \leftarrow & & & & & & & & & & 0 \\ E_{5,5} & & E_{5,4} & & E_{5,3} & & E_{5,2} & & E_{5,1} & & & & & & & & 5 \\ & & \dots & & & & & & & & & & & & & & 5 \\ & & & & & & & & & & & & & & & & 4 \\ & & & & & & & & & & & & & & & & 2 \\ & & & & & & & & & & & & & & & & 2 \\ & & & & & & & & & & & & & & & & 0 \end{array} \iff \begin{array}{cccccccccccccccc} & & & & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & & & 1 \\ & & & & & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & & & & & 5 \\ & & & & & & & & & & & & & & & & 5 \\ & & & & & & & & & & & & & & & & 4 \\ & & & & & & & & & & & & & & & & 2 \\ & & & & & & & & & & & & & & & & 2 \\ & & & & & & & & & & & & & & & & 0 \end{array}$$

The above scheme was later rediscovered several times in the literature (see [Kem33, MSY96]). A recent survey on down-up permutations and Euler numbers is given by Stanley [Sta09].

A sequence of sets $(X_{n,k})_{1 \leq k \leq n}$ is called an *Entringer family* if the cardinality of $X_{n,k}$ is equal to $E_{n,k}$ for $1 \leq k \leq n$.

Let $X = \{x_1, \dots, x_n\}_<$ be an ordered set such that $x_1 < \dots < x_n$. An *increasing tree* on X is a spanning tree of the complete graph on X , rooted at x_1 and oriented from the smallest vertex x_1 , such that the vertices increase along the edges. Let \mathcal{BT}_n be the set of *binary increasing trees* T on $[n]$, i.e., the increasing trees such that at most two edges go out from every vertex (see Figure 1).

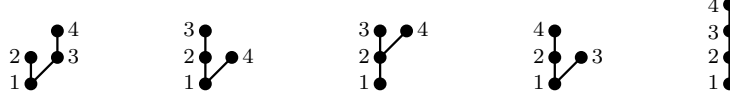


FIGURE 1. The binary increasing trees on $[4]$

Foata and Schützenberger proved in [FS73, §5] that the Euler number E_n is the cardinality of \mathcal{BT}_n . A one-to-one correspondence between \mathcal{DU}_n and \mathcal{BT}_n was then constructed by Donaghey [Don75] (see also [Cal05]). However the tree counterpart of Entringer’s result was found only in 1982 by Poupard [Pou82]. If T is a binary increasing tree and if (i, j) is an edge in T , $i < j$, we call i the *parent* of j , and j a *child* of i . If i has no child, we say that i is a *leaf* of T . A *path* in T is a sequence of vertices (a_i) such that a_i is a child of a_{i-1} in T , and the *minimal path* of T is the path $(a_i)_{1 \leq i \leq \ell}$ such that $a_1 = 1$, a_i ($i = 2, \dots, \ell$) is the smallest child of a_{i-1} and a_ℓ is a leaf, denoted by $p(T)$. Let’s denote by $\mathcal{BT}_{n,k}$ the set of trees $T \in \mathcal{BT}_n$ such that $p(T) = k$.

Theorem 1.2 (Poupard). *The sequence $(\mathcal{BT}_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Note that contrary to the case of down-up permutations, it is not easy to interpret recurrence (1) in the model of binary increasing increasing trees. Indeed, Donaghey’s bijection doesn’t induce a bijection between $\mathcal{DU}_{n,k}$ and $\mathcal{BT}_{n,k}$ and Poupard’s proof in [Pou82] was analytic in nature. Finding a direct explanation in the model of trees was then raised as an open problem in [KPP94]. The first aim of this paper is to build a bijection between $\mathcal{DU}_{n,k}$ and $\mathcal{BT}_{n,k}$ and answer the above open problem. In other words, we have the following theorem.

Theorem 1.3. *For all $n \geq 1$, there is an explicit bijection $\Psi : \mathcal{DU}_n \rightarrow \mathcal{BT}_n$ satisfying*

$$\forall \pi \in \mathcal{DU}_n, \quad \text{FIRST}(\pi) = \text{LEAF}(\Psi(\pi)),$$

where $\text{FIRST}(\pi)$ is the first element of the permutation π and $\text{LEAF}(\Psi(\pi))$ is the leaf of the minimal path of the tree $\Psi(\pi)$.

Poupard [Pou82, Pou97] gave also other interpretations for Entringer numbers $E_{n,k}$ (see Section 4) in binary increasing trees and down-up permutations with induction proofs. Our second aim is to provide simple bijections between the other interpretations of Poupard in down-up permutations and the original interpretation in $\mathcal{DU}_{n,k}$. Note that some other interpretations of Entringer numbers $E_{n,k}$ in the model of increasing trees were given in [KPP94]. Recently, two new interpretations of Euler numbers were given by Martin and Wagner [MW09] in the model of G-words and R-words. We shall give the corresponding interpretations of the Entringer number $E_{n,k}$ in the later models.

The rest of this paper is organized as follows. In Section 2, we introduce an intermediate model $\mathcal{ES}_{n,k}$ and present a bijection ψ between $\mathcal{DU}_{n,k}$ and $\mathcal{ES}_{n,k}$. In Section 3, we describe a bijection φ between $\mathcal{ES}_{n,k}$ and $\mathcal{BT}_{n,k}$ so that $\Psi = \varphi \circ \psi$ provides the bijection for Theorem 1.3. As an application, in Subsection 3.2, we give a direct interpretation of (1) in the

model of increasing trees. In Section 4, we recall the other interpretations of $E_{n,k}$ found by Poupard and establish simple bijections between these models. In Section 5, we give some new interpretations for $E_{n,k}$, first refining the results of Martin and Wagner [MW09] in their model of G-words and R-words, and secondly introducing the new model of U-words.

2. THE LEFT-TO-RIGHT CODING ψ OF DOWN-UP PERMUTATIONS

Consider down-up permutations on any finite subset $I = \{a_1, a_2, \dots, a_m\}_<$ of \mathbb{N} . Two elements a and b in I are said to be *adjacent* if there is no $c \in I$ between a and b . Let π be a down-up permutation on I , i.e., $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \dots$. Suppose $\pi_1 = a_i$ and $\pi_2 = a_j$ with $a_i > a_j$. If π_1 and π_2 are adjacent, then, deleting $\pi_1\pi_2$, we obtain again a down-up permutation on $I \setminus \{\pi_1, \pi_2\}$, otherwise, we can apply successively the adjacent transpositions $(a_i, a_{i-1}), (a_{i-1}, a_{i-2}), \dots, (a_{j+2}, a_{j+1})$ to π (from left-to-right):

$$\pi^{(1)} = (a_i, a_{i-1}) \circ \pi, \quad \pi^{(2)} = (a_{i-1}, a_{i-2}) \circ \pi^{(1)}, \quad \dots, \quad \pi^{(i-j-1)} = (a_{j+2}, a_{j+1}) \circ \pi^{(i-j-2)},$$

so that all the permutations $\pi^{(1)}, \dots, \pi^{(k-j-1)}$ are down-up permutations and the first two elements in $\pi^{(i-j-1)}$ are adjacent. Deleting the first two elements, we get again a down-up permutation, say $\pi^{(i-j)}$, on $I \setminus \{a_{j+1}, a_j\}$. If we register (a, b) for the composition from left with the adjacent involution (a, b) , and $(a, b)^*$ for the deletion of the first two letters a and b , then the operations in the above process can be encoded by the word

$$(a_i, a_{i-1})(a_{i-1}, a_{i-2}) \dots (a_{j+2}, a_{j+1})(a_{j+1}, a_j)^*.$$

Since the resulting permutation $\pi^{(i-j)}$ is still down-up, we can iterate this process until we obtain the empty permutation. Clearly the last deletion is $(n)^*$ if n is odd. We shall call *left-to-right code* the resulting sequence of the successive operations in this process and denote it by $\psi(\pi) = (\Delta_\ell)_\ell$, where each entry Δ_ℓ is either a transposition (j, i) , a deletion $(j, i)^*$, $1 \leq i < j \leq n$, or the deletion $(n)^*$. Formally, we can write the algorithm as follows:

- (1) Start with $(\pi, \Delta = \emptyset)$ and support set $I = \{a_1, a_2, \dots, a_m\}_<$
- (2) While $\text{Card}(I) \geq 2$, do:
 - (a) While there is $a \in I$ such that $\pi_1 > a > \pi_2$, do:
$$\Delta \leftarrow (\Delta, (\pi_1, a')), \text{ where } a' = \max\{a \in I \mid \pi_1 > a > \pi_2\},$$

$$\pi \leftarrow (\pi_1, a') \circ \pi.$$
 - (b) If there is no $a \in I$ such that $\pi_1 > a > \pi_2$, do:
$$\Delta \leftarrow (\Delta, (\pi_1, \pi_2)^*),$$

$$\pi \leftarrow \pi_3\pi_4 \dots \pi_n \text{ (eventually } \pi = \emptyset),$$

$$I \leftarrow I \setminus \{\pi_1, \pi_2\}.$$
- (3) If $\text{Card}(I) = 1$ with $I = \{a_m\}$, do:
$$D_\pi \leftarrow (\Delta, (a_m)^*),$$

$$\pi \leftarrow \emptyset,$$

$$I \leftarrow \emptyset.$$

Example 2.1. If $\pi = 748591623 \in \mathcal{DU}_{9,7}$, then the algorithm goes as follows:

Step	$\pi^{(\ell)}$	Δ_ℓ
0	748591623	\emptyset
1	648591723	$(7, 6)$
2	548691723	$(6, 5)$
3	8691723	$(5, 4)^*$
4	7691823	$(8, 7)$
5	91823	$(7, 6)^*$
6	81923	$(9, 8)$
7	31928	$(8, 3)$
8	21938	$(3, 2)$
9	938	$(2, 1)^*$
10	839	$(9, 8)$
11	9	$(8, 3)^*$
12	\emptyset	$(9)^*$

Thus, the left-to-right code of π is

$$\psi(\pi) = (7, 6) (6, 5) (5, 4)^* (8, 7) (7, 6)^* (9, 8) (8, 3) (3, 2) (2, 1)^* (9, 8) (8, 3)^* (9)^*.$$

A *domino* on $[n]$ is an ordered pairs (j, i) ($1 \leq i < j \leq n$) and a *starred domino* on $[n]$ is a starred ordered pairs $(j, i)^*$ ($1 \leq i < j \leq n$) or $(n)^* = (n, n)^*$. Let \mathbb{A}_n be the alphabet consisting of dominos (starred or non) on $[n]$.

Definition 2.2. A word $\Delta = \Delta_1 \dots \Delta_r$ on \mathbb{A}_n is an *encoding sequence* of $[n]$ if the following conditions are verified:

- (i) the entries of starred dominos are disjoint and their union equals $[n]$,
- (ii) if $\Delta_\ell = (j, i)^*$, then the next domino (if there is one) starts with an entry $> i$, and no entry of a later domino lies between i and j ,
- (iii) if $\Delta_\ell = (j, i)$, then both i and j appear in a later domino, with i the first entry of the next domino, and each integer between i and j appears in an earlier starred domino.

Remark 2.3. It is clear from the definition that $(n, i)^*$ ($1 \leq i \leq n$) can only take the last position in an encoding sequence and an encoding sequence must start with $(k, k-1)$ or $(k, k-1)^*$ for $2 \leq k \leq n$.

We denote by \mathcal{ES}_n the set of encoding sequences of $[n]$, and by $\mathcal{ES}_{n,k}$ the subset of \mathcal{ES}_n consisting of encoding sequences starting with $(k, k-1)$ or $(k, k-1)^*$, $2 \leq k \leq n$. For example, the set \mathcal{ES}_4 is the union of the three subsets:

$$\begin{aligned} \mathcal{ES}_{4,2} &= \{(2, 1)^* (4, 3)^*\}, \\ \mathcal{ES}_{4,3} &= \{(3, 2)^* (4, 1)^*, (3, 2) (2, 1)^* (4, 3)^*\}, \\ \mathcal{ES}_{4,4} &= \{(4, 3) (3, 2)^* (4, 1)^*, (4, 3) (3, 2) (2, 1)^* (4, 3)^*\}. \end{aligned}$$

Theorem 2.4. For all $n \geq 1$ and $k \in [n]$, the mapping $\psi : \mathcal{DU}_{n,k} \rightarrow \mathcal{ES}_{n,k}$ is a bijection. Therefore, the sequence $(\mathcal{ES}_{n,k})_{1 \leq k \leq n}$ is an Entringer family.

Proof. Let $\pi = \pi_1 \dots \pi_n$ be an element in $\mathcal{DU}_{n,k}$. Then $\pi_1 = k$, so the first letter of $\psi(\pi)$ is (k, i) or $(k, i)^*$ ($1 \leq i < k$) by definition of ψ . It remains to show that the word $\psi(\pi)$ verifies the conditions (i)-(iii) of Definition 2.2. Since the process reduces the permutation π to empty permutation, the condition (i) is verified.

- If $\Delta_\ell = (j, i)^*$, as i and j are adjacent in the support set of $\pi^{(\ell)}$, the integers between i and j have been removed in previous starred dominos, also the first entry of the next domino is $> i$ because $\pi^{(\ell)}$ is down-up.
- If $\Delta_\ell = (j, i)$, as i and j are adjacent in the support set of $\pi^{(\ell)}$, the integers between i and j have been removed in previous starred dominos, also the next domino must be (i, m) or $(i, m)^*$ with $i > m$ because i is the first entry of $\pi^{(\ell)}$.

It results that $\psi(\pi) \in \mathcal{ES}_{n,k}$.

Conversely, starting from an encoding sequence $\Delta = \Delta_1 \dots \Delta_\ell \in \mathcal{ES}_{n,k}$, we construct by induction $\pi^{(j)}$ such that $\text{FIRST}(\pi^{(j)})$ equals the first entry of Δ_j for $j = \ell, \ell - 1, \dots, 1$.

First, if $\Delta_\ell = (n)^*$ then define $\pi^{(\ell)} = n$, if $\Delta_\ell = (n, i)^*$ with $i < n$, then define $\pi^{(\ell)} = n i$.

Assume that $\pi^{(j+1)}$ is constructed with $\text{First}(\pi^{(j+1)}) = k_{j+1}$. By definition of Δ , there are two cases:

- if $\Delta_j = (k_j, k_{j+1})$, where k_j and k_{j+1} are adjacent in the support set of $\pi^{(j+1)}$, then define $\pi^{(j)} := (k_j, k_{j+1}) \circ \pi^{(j+1)}$. This permutation is still down-up and the first element of $\pi^{(j)}$ is k_j ;
- if $\Delta_j = (a_j, b_j)^*$, where $a_j > b_j < k_{j+1}$, and a_j, b_j are not in the support set of $\pi^{(j+1)}$, then define $\pi^{(j)}$ as the word $a_j b_j \pi^{(j+1)}$. Since $a_j > b_j < k_{j+1}$, the permutation $\pi^{(j)}$ is down-up with a_j as the first element.

Let $\psi^{-1}(\Delta) := \pi^{(1)}$, which is an element in $\mathcal{DU}_{n,k}$. ■

Remark 2.5. Denote the largest integer less than x by $\lfloor x \rfloor$ and the number of ordered pairs $(i, j) \in \{1, \dots, n\}$ such that $i + 1 < j$ and $\pi_i > \pi_{i+1} < \pi_j < \pi_i$ by $(31-2)\pi$. Then, one can show that the length of the sequence $\psi(\pi)$ is equal to

$$(31-2)\pi + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Indeed, $(31-2)\pi$ corresponds to the number of occurrences of terms (j, i) , $j > i$, in $\psi(\pi)$, and there are $\left\lfloor \frac{n+1}{2} \right\rfloor$ occurrences of terms $(j, i)^*$, $j > i$, in $\psi(\pi)$. Note that various formulae for counting 31-2-patterns in down-up permutations are given in [Che08, JV10, SZ10].

Proposition 2.6. *Let $n \geq 2$ and $k \geq 2$. The number of elements starting with $(k, k-1)$ equals $E_{n,k-1}$, and the number of elements starting with $(k, k-1)^*$ equals $E_{n-1, n+1-k}$.*

Proof. Let $\Delta \in \mathcal{ES}_{n,k}$. If $\Delta_1 = (k, k-1)$, the remaining sequence $(\Delta_2, \Delta_3, \dots)$ is still an encoding sequence of $[n]$, starting with $\Delta_2 \in \{(k-1, i), (k-1, i), 1 \leq i \leq k-2\}$. Thus, there are $E_{n,k-1}$ encoding sequences starting by $(k, k-1)$. If $\Delta_1 = (k, k-1)^*$, the remaining sequence $(\Delta_2, \Delta_3, \dots)$ doesn't contain the elements k and $k+1$ and starts with an element in $\{(i, j), (i, j)^*, 1 \leq j \leq i-1\}$ with $i \geq k+1$. In other words, this is an encoding sequence of $n-2$ elements, starting with an integer i that must be greater than the $k-2$ first elements. Thus, there are $E_{n-2, k-1} + E_{n-2, k} + \dots + E_{n-2, n-2} = E_{n-1, n+1-k}$ encoding sequences starting by $(k, k-1)^*$. ■

Since any sequence in $\mathcal{ES}_{n,k}$ begins with either $(k, k-1)$ or $(k, k-1)^*$ ($2 \leq k \leq n$), Entringer's formula (1) results from the above proposition.

3. THE LEFT-TO-RIGHT CODING OF BINARY TREES

3.1. The bijection $\varphi : \mathcal{ES}_{n,k} \rightarrow \mathcal{BT}_{n,k}$. Starting from an encoding sequence $\Delta = \Delta_1 \dots \Delta_\ell \in \mathcal{ES}_{n,k}$, we construct a tree $T = \varphi(\Delta) \in \mathcal{BT}_{n,k}$ by reading the sequence Δ in reverse order, i.e., from right to left. More precisely, for $m = \ell, \ell - 1, \dots, 1$, we shall construct a tree T_m corresponding to the word $\Delta_m \dots \Delta_{\ell-1} \Delta_\ell$ such that

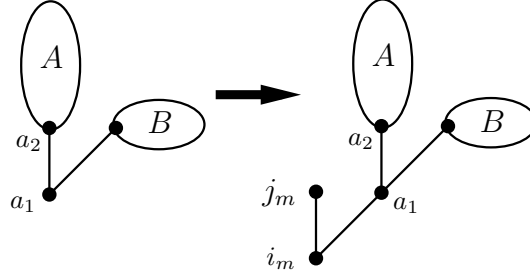
$$\Delta_m = (j_m, i_m) \text{ or } (j_m, i_m)^* \implies \text{LEAF}(T_m) = j_m, \quad (2)$$

and define $T = T_1 := \varphi(\Delta)$. The algorithm goes as follows:

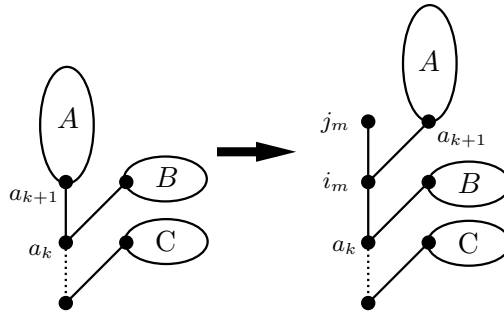
If $\Delta_\ell = (n)^*$, construct the tree T_ℓ with only one vertex n ; if $\Delta_\ell = (n, i)^*$, construct the increasing tree T_ℓ with only one edge $i \rightarrow n$. Clearly (2) is verified.

Assume that we have constructed such a tree T_{m+1} corresponding to the word $\Delta_{m+1} \dots \Delta_\ell$.

- (i) If $\Delta_m = (j_m, i_m)^*$, we add vertices i_m and j_m to the tree T_{m+1} to obtain T_m . Suppose that the minimal path of T_{m+1} is (a_1, \dots, a_{p_m}) .
 - If $i_m < a_1$, add the edges (i_m, a_1) and (i_m, j_m) to the tree T_{m+1} . Then, the tree T_m is an increasing tree rooted at i_m with (i_m, j_m) as the minimal path.

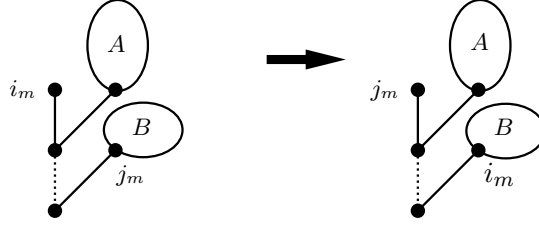


- If $i_m > a_1$, by induction hypothesis and property (ii) of encoding sequences, we see that $a_1 < m$. Hence, there exists $k \in \{1, \dots, p_m - 1\}$ such that $a_k < i_m < a_{k+1}$. Then, erase the edge (a_k, a_{k+1}) , create the edges (a_k, i_m) , (i_m, a_{k+1}) and (i_m, j_m) . Clearly, the tree T_m is an increasing tree with (i_m, j_m) as the last edge of the minimal path.

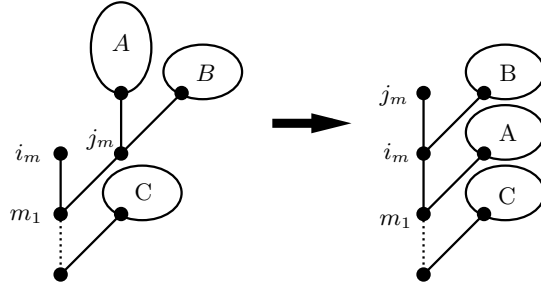


- (ii) If $\Delta_m = (j_m, i_m)$, where i_m and j_m are not siblings in T_{m+1} , by induction hypothesis and property (iii) of encoding sequences, we derive that i_m is at the end of the minimal path. Then, we transform the tree T_{m+1} as follows: just exchange the places of i_m and j_m in T_{m+1} . The tree remains increasing because Then j_m is at the end of the

minimal path in T_m .



- (iii) If $\Delta_m = (j_m, i_m)$, where i_m and j_m are siblings in T_{m+1} , as in the previous case, i_m is at the end of the minimal path. Then, transform T_{m+1} with the following procedure. If m_1 denotes the parent of i_m and j_m in T , erase the edge (m_1, j_m) , create an edge (i_m, j_m) , then if A and B are the two subtrees starting from j_m with $\min(A) < \min(B)$ (eventually B is empty), cut the subtree A from j_m and add it as a direct subtree of m_1 , cut the subtree B from j_m and add it as a direct subtree of i_m . The procedure can be illustrated with the following picture:



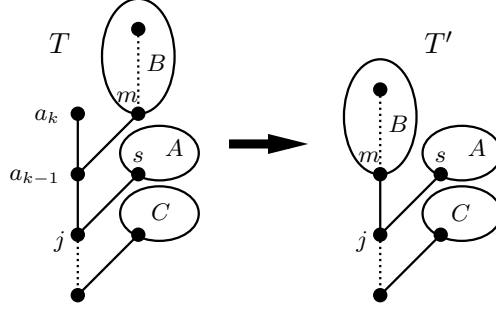
Let $\varphi(\Delta) := T_1$, which is an element in $\mathcal{BT}_{n,k}$.

Theorem 3.1. *For all $n \geq 1$ and $k \in [n]$, the mapping $\varphi : \mathcal{ES}_{n,k} \longrightarrow \mathcal{BT}_{n,k}$ is a bijection.*

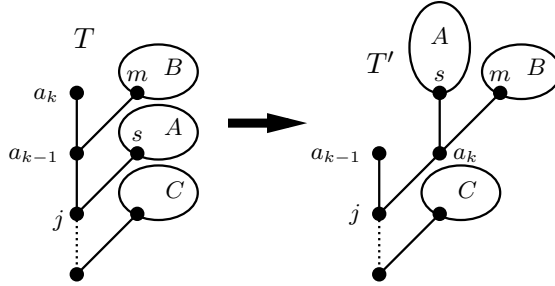
Proof. It is sufficient to construct the inverse mapping of φ to show that this is a bijection. Given T an increasing tree on the ordered set $\{a_1, \dots, a_n\}$ with $a_1 < \dots < a_n$, such that $p(T) = a_k$ (that can be interpreted by an element of $\mathcal{BT}_{n,k}$), we construct an encoding sequence $\Delta = \varphi^{-1}(T)$ of $[n]$ recursively as follows:

- (a) If a_{k-1} is the parent of a_k in T , then let m ($m > a_k$) be the other child of a_{k-1} ($m = \infty$ if a_k is the only child of a_{k-1}) and s ($s > k$) be a sibling of a_{k-1} ($s = \infty$ if a_{k-1} has no sibling), and j the parent of a_{k-1} in T .
 - (a1) If $m < \infty$ and $m < s$, then define $\varphi^{-1}(T) = ((a_k, a_{k-1})^*, \varphi^{-1}(T'))$, where T' is the tree obtained from T by deleting the vertices a_{k-1} , a_k and their adjacent

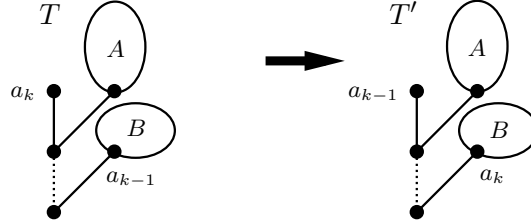
edges in T , and adding a new edge between m and j .



- (a2) In the other cases ($m = \infty$ or $m > s$), then define $\varphi^{-1}(T) = ((a_k, a_{k-1}), \varphi^{-1}(T'))$, where T' is the tree obtained from T by erasing the edges (a_{k-1}, a_k) , (a_{k-1}, m) and (j, s) in T , and adding the edges (j, a_k) , (a_k, s) , (a_k, m) . The procedure can be illustrated with the following picture:



- (b) If a_{k-1} is not the parent of a_k in T , then define $\varphi^{-1}(T) = ((a_k, a_{k-1}), \varphi^{-1}(T'))$, where T' is the tree obtained from T by exchanging the labels a_{k-1} and a_k in T .



Note that cases (a1), (a2) and (b) in the construction of φ^{-1} correspond, respectively, to cases (i), (ii) and (iii) of the construction of φ .

It remains to prove that the obtained sequence Δ verifies the points (i)-(iii) of Definition 2.2.

- It is easily seen that each integer of $[n]$ is removed once off T . So (i) is verified.
- If an element $(j, i)^*$ appears in Δ , that corresponds to the case (a1), when we delete the vertices i and j from the tree T . Then the next elements in Δ don't contain either i or j since they correspond to $\varphi^{-1}(T')$. Moreover, if we are in the case (a1), the minimal path in the tree T' contains at least one element m with $m > j > i$, so the next element in Δ must be (m, k) with $m > k$. Thus (ii) is verified.
- If an element (j, i) appears in Δ , in both Case (a2) or Case (b), the tree T' has i as the leaf of the minimal path. Then, the next element in Δ must be (i, k) with

$i > k$. Moreover, i and j must be consecutive elements in the ordered set of labels in T . Then the elements ℓ such that $i < \ell < j$ don't appear in T . Thus (iii) is verified. ■

Let $\Psi = \varphi \circ \psi$. Then $\Psi : \mathcal{DU}_{n,k} \rightarrow \mathcal{BT}_{n,k}$ is a bijection satisfying $\pi_1 = p(\Psi(\pi))$ for all $\pi \in \mathcal{DU}_{n,k}$. Thus Theorem 1.3 is proved.

Example 3.2. Continuing the Example 2.1, we apply Ψ to π by using the known LR-code of $\pi = 748591623$. The details are given in Figure 2.




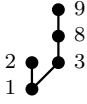



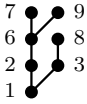
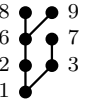
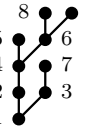
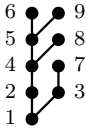
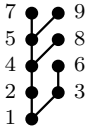
$\pi^{(m)}$	9	839	938	21938	31928	81923
$\text{FIRST}(\pi^{(m)})$	9	8	9	2	3	8
Δ_m	$(9)^*$	$(8, 3)^*$	$(9, 8)$	$(2, 1)^*$	$(3, 2)$	$(8, 3)$
$T_m = \Psi(\pi^{(m)})$						
$\text{LEAF}(T_m)$	9	8	9	2	3	8
$\pi^{(m)}$	91823	7691823	8691723	548691723	648591723	748591623
$\text{FIRST}(\pi^{(m)})$	9	7	8	5	6	7
Δ_m	$(9, 8)$	$(7, 6)^*$	$(8, 7)$	$(5, 4)^*$	$(6, 5)$	$(7, 6)$
$T_m = \Phi(\pi^{(m)})$						
$\text{LEAF}(T_m)$	9	7	8	5	6	7

FIGURE 2. The construction of the tree $\Psi(748591623)$

3.2. Interpretation of Entringer's formula in \mathcal{BT}_n . Following the interpretation of (1) in \mathcal{ES}_n (cf Remark 2.6) and the bijection φ , we must consider the decomposition of the set $\mathcal{BT}_{n,k}$. The first step in the construction of φ^{-1} would consist in either removing the elements $k-1$ and k , or the first step transforms the tree to obtain another tree of \mathcal{BT}_n .

For T an element of $\mathcal{BT}_{n,k}$, we say that the edge $(k-1, k)$ is *removable* if $k-1$ is the parent of k and if $k-1$ has another child m that is not greater than the sibling of $k-1$ (if such a sibling exists). For a visual representation, a tree T has its edge $(k-1, k)$ removable if it corresponds to the case A-1 in the proof of Theorem 3.1.

If the edge $(k-1, k)$ is not removable, the tree obtained after the first operation in the construction of φ^{-1} will be an increasing tree with n elements such that $k-1$ is the leaf of the main chain. Then, there are exactly $E_{n,k-1}$ trees such that the edge $(k-1, k)$ is not removable.

If the edge is removable, the tree obtained with the first operation in the construction of φ^{-1} will be an increasing tree with $n-2$ elements (without the elements $k-1$ and k), and the end of the minimal path must be an element i greater than the $k-2$ first elements. Thus,

there are $E_{n-2,k-1} + E_{n-2,k} + \dots + E_{n-2,n-2} = E_{n-1,n-k+1}$ increasing trees such that the edge $(k-1, k)$ is removable.

Finally, an interpretation of (1) appears in the model of \mathcal{T}_n . The decomposition according to the removability of the edge $(k-1, k)$ in $T \in \mathcal{BT}_{n,k}$ gives (1).

4. POUPARD'S OTHER ENTRINGER FAMILIES

4.1. Another interpretation in increasing trees. Let $\mathcal{BT}'_{n,k}$ be the set of trees $T \in \mathcal{BT}_n$ such that the parent of n in T is $k-1$. By using recurrence relations Poupard proved that $E_{n,k}$ is also the number of trees in $\mathcal{BT}'_{n,k}$. A bijection φ' between $\mathcal{BT}_{n,k}$ and $\mathcal{BT}'_{n,k}$ was given in [KPP94, §6] for a more general class of increasing trees that they call geometric.

4.2. Another interpretation in down-up permutations. If π is a permutation of $\mathcal{DU}_{n,k}$, define $\theta(\pi)$ as follows:

- if $k < n - k + 1 + \pi_2$, then $\theta(\pi) = (n - k + 1 + \pi_2, n - k + \pi_2, \dots, k + 1, k) \circ \pi$,
- if $k > n - k + 1 + \pi_2$, then $\theta(\pi) = (n - k + 1 + \pi_2, n - k + 2 + \pi_2, \dots, k - 1, k) \circ \pi$.

Since π is down-up, $\pi_2 < k = \pi_1$. If $k < n - k + 1 + \pi_2$, π_2 is unchanged by the cycle and then $\sigma(\pi)_2 = \pi_2$. Thus $\sigma(\pi)_2 < k < n - k + 1 + \pi_2 = \sigma(\pi)_1$ and $\theta(\pi)$ is still down-up. If $k > n - k + 1 + \pi_2$, since $k \leq n$, then $n - k + 1 + \pi_2 \geq \pi_2 + 1$, so π_2 is unchanged by the cycle, $\sigma(\pi)_1 = n - k + 1 + \pi_2 > \pi_2 = \sigma(\pi)_2$ and $\theta(\pi)$ is still down-up.

Let's denote by $\mathcal{DU}'_{n,k}$ the set of permutations $\pi \in \mathcal{DU}_n$ such that $\pi_1 - \pi_2 = n + 1 - k$.

Theorem 4.1. *For all $n \geq 1$ and $k \in [n]$, the mapping θ is a bijection from $\mathcal{DU}_{n,k}$ to $\mathcal{DU}'_{n,k}$. Moreover, for every $\pi \in \mathcal{DU}_{n,k}$, we have $\theta(\pi)_2 = \pi_2$.*

Proof. By construction, the mapping θ is clearly invertible. Moreover, for $\sigma \in \mathcal{DU}_n$ with $\sigma_1 - \sigma_2 = n - k + 1$,

- if $k < n - k + 1 + \sigma_2$, then $\theta^{-1}(\sigma) = (k, k + 1, \dots, n - k + \sigma_2, n - k + 1 + \sigma_2) \circ \sigma$,
- if $k > n - k + 1 + \sigma_2$, then $\theta^{-1}(\sigma) = (k, k - 1, \dots, n - k + 2 + \sigma_2, n - k + 1 + \sigma_2) \circ \sigma$,

thus $\theta^{-1}(\sigma) \in \mathcal{DU}_{n,k}$. ■

With Theorem 4.1, the following interpretation of Poupard, proved in [Pou97] by recurrence relations, can be recovered.

Corollary 4.2. *The sequence $(\mathcal{DU}'_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Since $\mathcal{DU}'_{n,k} \subset \mathcal{DU}_n$, we can define $\theta^2(\pi)$ for $\pi \in \mathcal{DU}_n$. Actually, it is easy to see that the mapping θ is an involution on \mathcal{DU}_n . The result can also be generalized with the following observation. For any $\pi \in \mathcal{DU}_n$, define the *complement permutation* $\bar{\pi}$ with $\bar{\pi}_i = n + 1 - \pi_i$ for $i \in [n]$. Denote by \mathcal{DU}_n^* the set of permutations π such that $\bar{\pi} \in \mathcal{DU}_n$.

Corollary 4.3. *For $n \geq 1$, we have*

$$\sum_{\pi \in \mathcal{DU}_n^*} q^{\pi_1} p^{\pi_2 - \pi_1} = \sum_{\pi \in \mathcal{DU}_n^*} p^{\pi_1} q^{\pi_2 - \pi_1}.$$

Proof. The mapping $\pi \mapsto \theta(\bar{\pi})$ is a bijection between $\{\pi \in \mathcal{DU}_n^* : \pi_1 = k\}$ and $\{\pi \in \mathcal{DU}_n^* : \pi_2 - \pi_1 = k\}$. Thus, the two statistics π_1 and $\pi_2 - \pi_1$ are equidistributed on \mathcal{DU}_n^* . Indeed, with proof of Theorem 4.1, $\pi \mapsto \theta(\bar{\pi})$ is a bijection between $\{\pi \in \mathcal{DU}_n^* : \pi_1 = k, \pi_2 - \pi_1 = \ell\}$ and $\{\pi \in \mathcal{DU}_n^* : \pi_1 = \ell, \pi_2 - \pi_1 = k\}$. Thus, the distribution of the two statistics is symmetric. ■

4.3. Interpretations in min-max alternating permutations. Recall that a permutation π on $[n]$ is an alternating permutation if $\pi_1 > \pi_2 < \pi_3 > \cdots$ or $\pi_1 < \pi_2 > \pi_3 < \cdots$. A *min-max alternating permutation* of $[n]$ is an alternating permutation in which 1 precedes n . For example, the min-max alternating permutations of $[4]$ are

$$1423, \quad 1324, \quad 3142, \quad 2314, \quad 2143.$$

Let \mathcal{MM}_n be the set of *min-max alternating permutations* of $[n]$. Denote by $\mathcal{MM}_{n,k}$ the set of $\pi \in \mathcal{MM}_n$ such that $|\pi_1 - \pi_2| = n+1-k$. The set $\mathcal{DU}'_{n,k}$ can be split in two disjoint subsets $\mathcal{DU}'_{n,k,1n}$ which is the set of permutations in $\mathcal{DU}'_{n,k} \cap \mathcal{MM}_n$ and $\mathcal{DU}'_{n,k,n1} = \mathcal{DU}'_{n,k} \setminus \mathcal{DU}'_{n,k,1n}$. If $\pi \in \mathcal{DU}'_{n,k,1n}$, define $\beta(\pi) = \pi$, and if $\pi \in \mathcal{DU}'_{n,k,n1}$, define $\beta(\pi) = \bar{\pi}$. Thus $\beta(\pi) \in \mathcal{MM}_n$ and $\beta(\pi)_1 - \beta(\pi)_2 = -(n+1-k)$.

Theorem 4.4. *For all $n \geq 1$ and $k \in [n]$, the mapping β is a bijection between $\mathcal{DU}'_{n,k}$ and $\mathcal{MM}_{n,k}$.*

With the previous theorem, the interpretation of Poupard, proved in [Pou97] by recurrence relations, can be recovered.

Corollary 4.5. *The sequence $(\mathcal{MM}_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Denote by $\mathcal{MM}'_{n,k}$ the set of $\pi \in \mathcal{MM}_n$ such that the term immediately before 1 is k , if $k \leq n-1$, and by $\mathcal{MM}'_{n,n}$ the set of $\pi \in \mathcal{MM}_n$ such that $\pi_1 = 1$.

We want to construct a bijection ρ between $\mathcal{DU}_{n,k}$ and $\mathcal{MM}'_{n,k}$.

If $k = n$, it suffices to define for $\pi \in \mathcal{DU}_{n,n}$, $\rho(\pi) = \bar{\pi}$. Then, $\rho(\pi) \in \mathcal{MM}'_{n,n}$.

Assume that $k \leq n-1$. The set $\mathcal{DU}_{n,k}$ can be split in two disjoint subsets $\mathcal{DU}_{n,k,1n}$ which is $\mathcal{DU}_{n,k} \cap \mathcal{MM}_n$ and $\mathcal{DU}_{n,k,n1} := \mathcal{DU}_{n,k} \setminus \mathcal{DU}_{n,k,1n}$. For an ordered set $I = \{a_1, \dots, a_n\}$ with $a_1 < \cdots < a_n$, denote by σ_I the permutation:

$$\sigma_I = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_{n-1} & \cdots & a_1 \end{pmatrix}$$

Then, for a permutation $\pi = \pi_1 \dots \pi_n$ on the ordered set I , denote by $\bar{\pi}$ the *complement permutation on the set I* , that is $\bar{\pi} := \sigma_I \circ \pi$, and π^R the *reverse permutation*:

$$\pi^R := \pi_n \pi_{n-1} \dots \pi_1.$$

Note that when $I = [n]$, the definition of the complement permutation coincides with the one in the Remark of Subsection 4.2.

Then, for a permutation $\pi \in \mathcal{DU}_{n,k}$,

- If $\pi \in \mathcal{DU}_{n,k,1n}$, we can write $\pi = \sigma_1 1 \sigma_2$. Then, define $\rho(\pi) = \sigma_1^R 1 \sigma_2$. Since $1 < \pi_1 > \pi_2$, $\rho(\pi)$ is still down-up, and the term just before 1 in $\rho(\pi)$ is $\pi_1 = k$.
- If $\pi \in \mathcal{DU}_{n,k,n1}$, we can write $\pi = \sigma_1 n \sigma_2$. Then, define $\rho(\pi) = \sigma_1^R 1 \bar{\sigma}_2$. Since $1 < \pi_1 > \pi_2$ and $\bar{\sigma}_2$ is down-up, $\rho(\pi)$ is still down-up, and the term just before 1 in $\rho(\pi)$ is $\pi_1 = k$.

Theorem 4.6. *For all $n \geq 1$ and $k \in [n]$, the mapping ρ is a bijection between $\mathcal{DU}_{n,k}$ and $\mathcal{MM}'_{n,k}$.*

Proof. In order to prove that ρ is a bijection, it suffices to describe the inverse of ρ . Let π be an element in \mathcal{MM}_n such that the term immediately before 1 is k . Following the construction of ρ , we have:

- If $\pi \in \mathcal{DU}_{n,k}$, write $\pi = \tau_1 1 \tau_2$. Then, $\rho^{-1}(\pi) = \tau_1^R 1 \tau_2$.

- If $\pi \notin \mathcal{DU}_{n,k}$, write $\pi = \tau_1 1 \tau_2$. Then, $\rho^{-1}(\pi) = \tau_1^R n \overline{\tau_2}$. ■

With the previous theorem, the following interpretation of Poupard, proved in [Pou97] by recurrence relations, can be recovered.

Corollary 4.7. *The sequence $(\mathcal{MM}'_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Denote by $\mathcal{MM}''_{n,k}$ the set of $\pi \in \mathcal{MM}_n$ such that the term immediately after n is $n+1-k$, if $k \leq n-1$, and $\mathcal{MM}''_{n,n}$ the set of $\pi \in \mathcal{MM}_n$ such that $\pi_n = n$.

Denote by ρ' the mapping defined for $\pi \in \mathcal{MM}''_{n,k}$ by $\rho'(\pi) = \overline{\pi^R}$.

Theorem 4.8. *For all $n \geq 1$ and $k \in [n]$, the mapping ρ' is a bijection between $\mathcal{MM}'_{n,k}$ and $\mathcal{MM}''_{n,k}$. Therefore, the sequence $(\mathcal{MM}''_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Proof. For $k \leq n-1$, $\pi \in \mathcal{MM}_n$ has k just before 1 if and only if $\rho'(\pi)$ has $n+1-k$ just after n . ■

5. NEW ENTRINGER FAMILIES

5.1. Interpretations in G-words and R-words. A permutation π of $I = \{a_1, \dots, a_n\}$ with $a_1 < \dots < a_n$ is called a *G-word* if

- (i) $\pi_1 = a_n, \pi_n = a_{n-1}$,
- (ii) $\pi_2 > \pi_{n-1}$ (if $n \geq 4$).

Similarly, a permutation π of I is called an *R-word* if previous conditions are satisfied when (ii) is replaced by

- (ii') $\pi_2 < \pi_{n-1}$ (if $n \geq 4$).

A G-word (resp. an R-word) is said to be *primitive* if for all $1 \leq i < j \leq n$, neither the word $\pi_i \pi_{i+1} \dots \pi_j$ nor the word $\pi_j \pi_{j-1} \dots \pi_i$ is a G-word (resp. an R-word). Denote respectively by \mathcal{GW}_n and \mathcal{RW}_n the set of primitive G-words on $[n+2]$ and primitive R-words on $[n]$. For examples, the G-words in \mathcal{GW}_4 are:

$$634215, \quad 642315, \quad 623415, \quad 643215, \quad 624315,$$

and the R-words in \mathcal{RW}_4 are:

$$621435, \quad 623145, \quad 614235, \quad 631245, \quad 624135.$$

These permutations were introduced in [Mar06] with the following problem. Let I_n be the ideal of all algebraic relations on the slopes of all lines that can be formed by placing n points in a plane. Then, under two orders, I_n is generated by monomials corresponding to respectively primitive G-words and primitive R-words.

Martin and Wagner proved [MW09] that E_n is the number of primitive G-words (resp. the number of primitive R-words) on $[n+2]$. Actually, this result can be refined to Entringer numbers by introducing a statistic on G-words.

Given a primitive G-word or an R-word π on $[n+2]$, the *route* of π is the sequence (α_i) defined by the following procedure:

- $\alpha_1 = n+2 (= \pi_1), \alpha_2 = n+1 (= \pi_{n+2}),$

- for $k \geq 2$, if $\alpha_k = \pi_i$, define $A_k = \{\alpha_1, \dots, \alpha_{k-1}\}$ and

$$\alpha_{k+1} = \begin{cases} \alpha_k & \text{if } \{\pi_{i-1}, \pi_{i+1}\} \subset A_k, \\ \max \left[\begin{array}{l} \{\pi_j | j < i \text{ and } \pi_j, \pi_{j+1}, \dots, \pi_{i-1} \notin A_k\} \\ \cup \{\pi_j | j > i \text{ and } \pi_{i+1}, \dots, \pi_{j-1}, \pi_j \notin A_k\} \end{array} \right] & \text{otherwise.} \end{cases}$$

One can represent the route of a G-word or an R-word π as a graph with the vertices $\pi_1, \pi_2, \dots, \pi_n$ ordered in a line, with only one path starting from n drawn upon the line and going successively, if it's possible, to $n-1, n-2, \dots, 1$ without crossings (see Figure 3 for an example). Denote $\mathcal{GW}_{n,k}$ (resp. $\mathcal{RW}_{n,k}$) the set of primitive G-words π on $[n+2]$ (resp. primitive R-words π on $[n+2]$) such that $\alpha_{n+2} = n+1-k$.

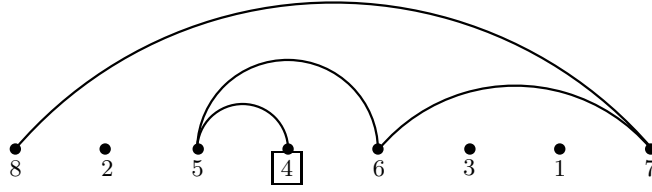


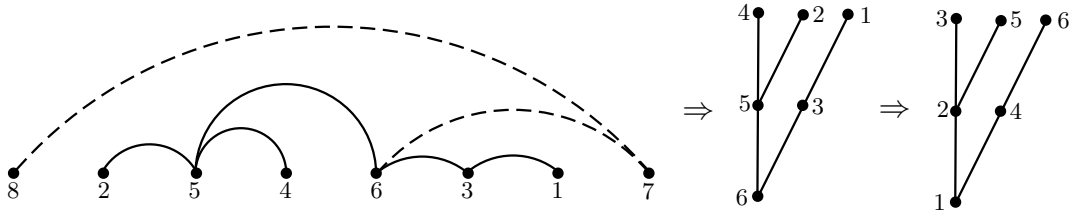
FIGURE 3. The route of the G-word $\pi = 82546317$

Theorem 5.1. *The sequences $(\mathcal{GW}_{n,k})_{1 \leq k \leq n}$ and $(\mathcal{RW}_{n,k})_{1 \leq k \leq n}$ are Entringer families.*

Proof. Use the bijection δ between \mathcal{GW}_n and \mathcal{BT}_n present in [MW09]. For π a primitive G-word on $\{a_1, \dots, a_{n+2}\}$ with $a_1 < \dots < a_{n+2}$, denote by π' the word $\pi_2 \dots \pi_{n+1}$. If π' is a word on $\{a_1, \dots, a_n\}$, with $a_1 < \dots < a_n$ and $a_n = \pi'_k$ for $k \in \{1, \dots, n\}$, define $T = \alpha(\pi')$ as the tree with root a_1 , from which two subgraphs go out, that are $\alpha(\pi'_1 \pi'_2 \dots \pi'_{k-1})$ and $\alpha(\pi'_{k+1} \pi'_{k+2} \dots \pi'_n)$ (eventually one of them or both are empty). The tree $\delta(\pi) = \alpha(\pi')$ is a binary increasing tree and the application δ is a bijection from \mathcal{GW}_n to \mathcal{BT}_n (see [MW09] for further details).

Moreover, it is easy to see that the labels upon the minimal path of $T = \delta(\pi)$ are successively $(n+1-a_1), (n+1-a_2), \dots, (n+1-a_m)$, where a_1, \dots, a_m ($a_1 > \dots > a_m$) are the different values that appear in the route of π . Thus, the leaf of the minimal path is k . Then, δ is a bijection between $\mathcal{GW}_{n,k}$ and $\mathcal{T}_{n,k}$.

For example, one can construct the tree that corresponds with the G-word $\pi = 82546317$ with this construction:



The analogous result for the R-word can be proved using the same method with the bijection δ' between \mathcal{RW}_n and \mathcal{BT}_n present in [MW09]. ■

5.2. Interpretations in U-words. We introduce here two new Entringer families.

Definition 5.2. A *U-word of length n* is a sequence $u = (u_i)_{1 \leq i \leq n}$ such that $u_1 = 1$ and $u_i + u_{i-1} \leq i$ for $i \in \{2, \dots, n\}$. We denote by \mathcal{UW}_n the set of U-words of length n .

For example, the U-words of length 4 are:

$$1111, \quad 1112, \quad 1113, \quad 1121, \quad 1122.$$

Denote by $\mathcal{UW}_{n,k}$ the set of U-words $(u_i) \in \mathcal{UW}_n$ such that $u_n = n + 1 - k$.

Theorem 5.3. *The sequence $(\mathcal{UW}_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Proof. For any finite set X , let $\#X$ denotes its cardinality. For $\pi \in \mathcal{DU}_{n,k}$, let $\gamma(\pi) = w^{RW}$, where $w = w_1 \dots w_n$ is the word defined by

$$w_i = \begin{cases} \#\{j \geq \pi_i, j \notin \{\pi_1, \pi_2, \dots, \pi_{i-1}\}\}, & \text{if } i \text{ is odd,} \\ \#\{j \leq \pi_i, j \notin \{\pi_1, \pi_2, \dots, \pi_{i-1}\}\}, & \text{if } i \text{ is even.} \end{cases}$$

For example, if $\pi = 6351724 \in \mathcal{DU}_{7,6}$, then the word w is computed as follows:

- $\{j \geq 6\} = \{6, 7\}$, so $w_1 = 2$,
- $\{j \leq 3, j \neq 6\} = \{1, 2, 3\}$, so $w_2 = 3$,
- $\{j \geq 5, j \notin \{3, 6\}\} = \{5, 7\}$, so $w_3 = 2$,
- $\{j \leq 1, j \notin \{3, 5, 6\}\} = \{1\}$, so $w_4 = 1$,
- $\{j \geq 7, j \notin \{1, 3, 5, 6\}\} = \{7\}$, so $w_5 = 1$,
- $\{j \leq 2, j \notin \{1, 3, 5, 6, 7\}\} = \{2\}$, so $w_6 = 1$,
- $\{j \geq 4, j \notin \{1, 2, 3, 5, 6, 7\}\} = \{4\}$, so $w_7 = 1$,

Then, $w = 2321111$ and $\gamma(\pi) = 1111232$.

We show that the mapping γ is a bijection between $\mathcal{DU}_{n,k}$ and $\mathcal{UW}_{n,k}$. Following the construction, $\gamma(\pi)_n = w_1 = n + 1 - \pi_1 = n + 1 - k$. Moreover, when $\gamma(\pi)_i = w_{n+1-i}$ is written, $n - i$ elements have been read in π before, thus the number of elements counted by $\gamma(\pi)_i$ must be less than i . Moreover, the numbers counted by $\gamma(\pi)_{i-1}$ and $\gamma(\pi)_i$ are in the $n - i$ elements that have not been read in π and are two disjoint sets since π is down-up. Thus $\gamma(\pi)_i + \gamma(\pi)_{i-1}$ must be less than i . Finally, $\gamma(\pi) \in \mathcal{UW}_{n,k}$.

Conversely, if $u \in \mathcal{UW}_{n,k}$, the permutation $\pi = \gamma^{-1}(u) \in \mathcal{DU}_{n,k}$ can be recovered with:

- $\pi_1 = n + 1 - u_n$
- $\forall n \geq 1, \pi_{2i}$ is the u_{n-2i+1} -st smallest element in $[n] \setminus \{\pi_1, \dots, \pi_{2i-1}\}$.
- $\forall n \geq 1, \pi_{2i+1}$ is the u_{n-2i} -st greatest element in $[n] \setminus \{\pi_1, \dots, \pi_{2i}\}$.

We are done. ■

Denote by $\mathcal{UW}'_{n,k}$ the set of U-words $(u_i) \in \mathcal{UW}_n$ such that $u_{n-1} + u_n = k$.

Theorem 5.4. *The sequence $(\mathcal{UW}'_{n,k})_{1 \leq k \leq n}$ is an Entringer family.*

Proof. There are two possibilities to prove this result.

Firstly, the mapping γ also induces a bijection between $\mathcal{DU}'_{n,k}$ and $\mathcal{UW}'_{n,k}$. For $\pi \in \mathcal{DU}'_{n,k}$, there exists $j \in [n]$ such that $\pi \in \mathcal{DU}_{n,j}$, so we can define $v = \gamma(\pi) \in \mathcal{UW}_{n,j} \subset \mathcal{UW}_n$. It suffices to show that $v \in \mathcal{UW}'_{n,k}$. In the construction of $\gamma(\pi)$, v_n is the number of elements that are greater than π_1 , and v_{n-1} is the number of elements that are less than π_2 . Then $v_n = n + 1 - \pi_1$ and $v_{n-1} = \pi_2$, and $v_{n-1} + v_n = n + 1 - (\pi_1 - \pi_2) = k$ since $\pi \in \mathcal{DU}'_{n,k}$.

Secondly, it is easy to construct a bijection $\alpha : \mathcal{UW}_{n,k} \longrightarrow \mathcal{UW}'_{n,k}$. For $u = (u_1, \dots, u_n) \in \mathcal{UW}_{n,k}$, let $\alpha(u) = (u_1, u_2, \dots, u_{n-1}, n + 1 - u_{n-1} - u_n)$. Since $u \in \mathcal{UW}_{n,k}$, $u_n - u_{n-1} \leq n + 1$,

so we have $\alpha(u) \in \mathcal{UW}_n$. Moreover, the last element $\alpha(u)_n = n + 1 - u_{n-1} - (n + 1 - k) = k - u_{n-1} = k - \alpha(u)_{n-1}$, so $\alpha(u) \in \mathcal{UW}'_{n,k}$. The mapping α is then clearly a bijection between $\mathcal{UW}_{n,k}$ and $\mathcal{UW}'_{n,k}$. \blacksquare

It follows immediately from the above theorems that the Euler number E_n is the number of U-words of length n for all integer $n \geq 1$.

6. CONCLUDING REMARKS

6.1. List of bijections for Entringer families. In what follows, we list all the twelve interpretations for Entringer families along with the bijections discussed in this paper:

- (1) the permutation $\pi \in \mathcal{DU}_{n,k}$ such that $\pi_1 = k$,
- (2) the encoding sequence $\Delta \in \mathcal{ES}_{n,k}$, obtained by $\Delta = \psi(\pi)$, where ψ is the bijection described in Section 2, then k is the first element read in Δ ,
- (3) the binary increasing increasing tree $T \in \mathcal{BT}_{n,k}$, obtained by $T = \varphi(\Delta)$, where φ is the bijection described in Section 3, then k is the leaf of the minimal path of T ,
- (4) the binary increasing increasing tree $T' \in \mathcal{BT}'_{n,k}$, obtained by $T' = \varphi'(T)$, where φ' is the bijection described in [KPP94, §6], then $k - 1$ is the parent of n in T' ,
- (5) the down-up permutation $\sigma \in \mathcal{DU}'_{n,k}$, obtained by $\sigma = \theta(\pi)$, where θ is the bijection described in Subsection 4.2, then $k = n + 1 - \sigma_1 + \sigma_2$,
- (6) the min-max alternating permutation $\sigma' \in \mathcal{MM}_{n,k}$, obtained by $\sigma' = \beta(\sigma)$, where β is the bijection described in Subsection 4.3, then $k = n + 1 - |\sigma_1 - \sigma_2|$,
- (7) the min-max alternating permutation $\tau_1 \in \mathcal{MM}'_{n,k}$, obtained by $\tau_1 = \rho(\pi)$, where ρ is the bijection described in Subsection 4.3, then k is the term immediately before 1 (or n if τ_1 starts with 1),
- (8) the min-max alternating permutation $\tau_2 \in \mathcal{MM}''_{n,k}$, obtained by $\tau_2 = \rho'(\tau_1)$, where ρ' is the bijection described in Subsection 4.3, then $n + 1 - k$ is the term immediately after n (or 1 if τ_2 ends with n),
- (9) the G-word $\pi' \in \mathcal{GW}_{n,k}$, obtained by $\pi' = \delta^{-1}(T)$, where δ is the bijection described in Subsection 5.1, then $n + 1 - k$ is the end of the route of π' ,
- (10) the R-word $\pi'' \in \mathcal{RW}_{n,k}$, obtained by $\pi'' = (\delta')^{-1}(T)$, where δ' is the bijection described in Subsection 5.1, then $n + 1 - k$ is the end of the route of π'' ,
- (11) the sequence $u \in \mathcal{UW}_{n,k}$, obtained by $u = \gamma(\pi)$, where γ is the bijection described in Subsection 5.2, then $n + 1 - k$ is the last element of u ,
- (12) the sequence $v \in \mathcal{UW}'_{n,k}$, obtained by $v = \gamma(\sigma) = \alpha(u)$, where α and γ are the bijections described in Subsection 5.2, then k is the sum of the two last elements of v .

We summarize the bijections of this paper in the diagram of Figure 4, where at the left we gather all the models in down-up permutations, and at the right we gather the models in the increasing trees.

6.2. Illustration for $n = 4$. In Figure 5, we summarize twelve interpretations for $E_{4,k}$, $k \in \{2, 3, 4\}$. In every column, the corresponding elements are described via the different bijections mentioned in the paper. Moreover, in the table, boxes point out the statistic $k = \pi_1$ if $\pi \in \mathcal{DU}_{n,k}$ and the corresponding statistics in the other models.

6.3. An open problem. Consider the so-called *reduced tangent numbers* $t_n = E_{2n+1}/2^n$. Poupard [Pou89] proved that t_n is the number of 0-2 increasing trees (i.e., the trees in \mathcal{BT}_n such that every vertex has 0 or 2 children). However, it seems that there is no interpretation

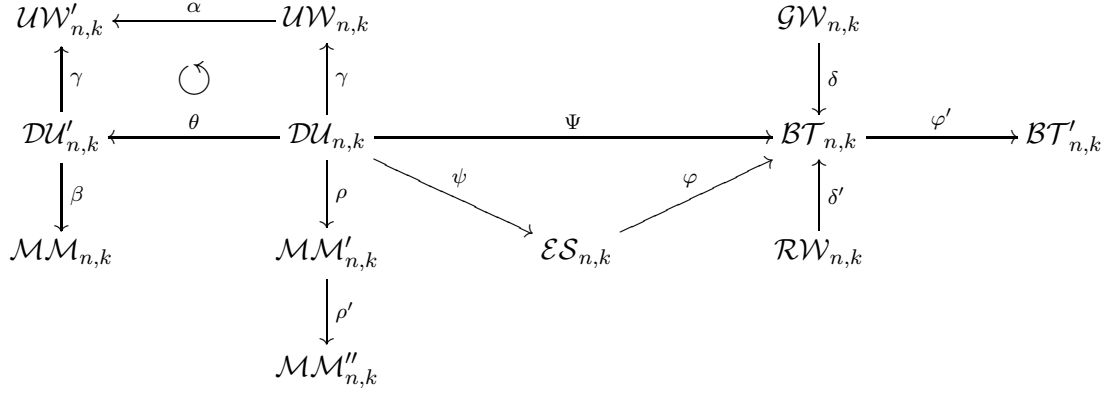


FIGURE 4. The bijections mentioned in the paper

à la André for t_n in down-up permutations. Furthermore, let $t_{n,k}$ denote the number of 0-2 increasing trees such that the leaf of the minimal path is k , then the sequence $(t_{n,k})$ is obviously a refinement of t_n as Entringer numbers are for Euler numbers.

Let s_n (resp. $s_{n,k}$) be the number of *split-pair arrangements* of $[n]$, that are arrangements σ of the multi-set $\{0, 0, 1, 1, 2, 2, \dots, n, n\}$ such that $\sigma(1) = n$ (resp. $\sigma(1) = \sigma(k+1) = n$) and, between the two occurrences of i in σ ($0 \leq i \leq n-1$), the number $i+1$ appears exactly once.

Recently, Graham and Zang [GZ08] proved that for $1 \leq k \leq n$, $s_{n,k} = t_{n,k}$. In particular, $s_n = t_n$. There is no bijective proof between Poupard's model and Graham and Zang's model.

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









	k	2	3		4	
(1)	$\pi \in \mathcal{DU}_{4,k}$	$\boxed{2}143$	$\boxed{3}241$	$\boxed{3}142$	$\boxed{4}231$	$\boxed{4}132$
(2)	$\Delta \in \mathcal{ES}_{4,k}$	$(\boxed{2}, 1)^*$ $(4, 3)^*$	$(\boxed{3}, 2)^*$ $(4, 1)^*$	$(\boxed{3}, 2)$ $(2, 1)^*$ $(4, 3)^*$	$(\boxed{4}, 3)$ $(3, 2)^*$ $(4, 1)^*$	$(\boxed{4}, 3)$ $(3, 2)$ $(2, 1)^*$ $(4, 3)^*$
(3)	$T \in \mathcal{BT}_{4,k}$					
(4)	$T' \in \mathcal{BT}'_{4,k}$					
(5)	$\sigma \in \mathcal{DU}'_{4,k}$	$\overbrace{41}^{\boxed{3}}32$	$\overbrace{42}^{\boxed{2}}31$	$\overbrace{31}^{\boxed{2}}42$	$\overbrace{32}^{\boxed{1}}41$	$\overbrace{21}^{\boxed{1}}43$
(6)	$\sigma' \in \mathcal{MM}_{4,k}$	$\overbrace{14}^{\boxed{3}}23$	$\overbrace{13}^{\boxed{2}}24$	$\overbrace{31}^{\boxed{2}}42$	$\overbrace{23}^{\boxed{1}}14$	$\overbrace{21}^{\boxed{1}}43$
(7)	$\tau_1 \in \mathcal{MM}'_{4,k}$	$\boxed{2}143$	$2\boxed{3}14$	$\boxed{3}142$	$\boxed{(4)}1324$	$\boxed{(4)}1423$
(8)	$\tau_2 \in \mathcal{MM}''_{4,k}$	$214\boxed{3}$	$14\boxed{2}3$	$314\boxed{2}$	$1324\boxed{(1)}$	$2324\boxed{(1)}$
(9)	$\pi' \in \mathcal{GW}_{4,k}$	$6\boxed{3}4215$	$64\boxed{2}315$	$6\boxed{2}3415$	$6432\boxed{1}5$	$6243\boxed{1}5$
(10)	$\pi'' \in \mathcal{RW}_{4,k}$	$6214\boxed{3}5$	$6\boxed{2}3145$	$614\boxed{2}35$	$63\boxed{1}245$	$624\boxed{1}35$
(11)	$u \in \mathcal{UW}_{4,k}$	$111\boxed{3}$	$112\boxed{2}$	$111\boxed{2}$	$112\boxed{1}$	$111\boxed{1}$
(12)	$v \in \mathcal{UW}'_{4,k}$	$11\overbrace{11}^{\boxed{2}}$	$11\overbrace{21}^{\boxed{3}}$	$11\overbrace{12}^{\boxed{3}}$	$11\overbrace{22}^{\boxed{4}}$	$11\overbrace{13}^{\boxed{4}}$

FIGURE 5. Twelve interpretations for $E_{4,k}$, $1 \leq k \leq 4$